Journal of Computational and Applied Mathematics 224 (2009) 405–414

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Period function monotonicity of planar vector fields[☆]

Khalil I.T. Al-Dosary [*]

Department of Mathematics, College of Arts and Sciences, University of Sharjah, P.O. Box 27272, Sharjah, United Arab Emirates

ARTICLE INFO

Article history:
Received 16 March 2008
Received in revised form 13 May 2008

MSC:
primary 34C25
34C05
34C07

Keywords:
Vector field
Center
Period function
Monotonicity
Isochronous centers

ABSTRACT

In this paper, we study planar differential systems possessing a center at the origin. We introduce functional conditions for centers and obtain sufficient conditions for monotonicity of period function and isochronicity of the center of systems of ODE reducible to the equation $\ddot{x} = f(x, \dot{x})$. These criteria are applied to obtain sufficient conditions for isochronicity of some special important systems.

© 2008 Elsevier B.V. All rights reserved.

1. Introduction

Consider the planar differential systems

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned} \tag{1.1}$$

with $P(x, y)$, $Q(x, y)$ functions of class C^1 defined in an open neighborhood U of the origin $(0, 0)$. To study the integrability problem of the vector field associated with system (1.1) we may investigate local first integrals. Assume the origin is a critical point of (1.1). We say that $(0, 0)$ is a center of (1.1) if it has a neighborhood W covered with nontrivial periodic solutions. Recall that system (1.1) has a center at the origin if and only if it has a Lyapounov first integral. When $(0, 0)$ is a center, we can define on $W\backslash\{(0, 0)\}$ the period function $T(x, y)$ which associates to every point $(x, y) \in W$ the minimal period of the periodic solution $\gamma_{(x,y)}$ passing through (x, y). T is obviously constant on periodic solutions, so that it is a first integral. We say that $(0, 0)$ is an isochronous center if the period function T is constant in a neighborhood of $(0, 0)$. The period function has been extensively studied by a number of different authors, see [1–9,11–17] and the references therein. There are many different methods of analysis have been applied to monotonicity of period function and isochronicity of centers. Methods applied in the study of isochronicity problem are, for instance, by linearization in [12], by reduction to second order differential equations in [11], by computation of isochronicity constants in [9], by the commutators method in [15,17], and by a transformation to the simple harmonic oscillator in [6]. In this paper we focus on the monotonicity of

☆ This paper is in final form and no version of it will be submitted for publication elsewhere.

* Tel.: +971 6 5050390; fax: +971 6 5050352.
E-mail address: dosary@sharjah.ac.ae.

0377-0427/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2008.05.021

the period function T of system (1.1), with critical point at the origin, in order to derive condition for isochronicity of the center. Some results given here are related to some results of [7], but with a completely different approach.

We will work with positive definite function

$$V_{(g,h)} : W_{(g,h)} \to R$$

defined as

$$V_{(g,h)}(x,y) = \int_0^x \frac{s}{g(s)}\mathrm{d}s + \int_0^y \frac{t}{h(t)}\mathrm{d}t \tag{1.2}$$

where g, h are continuous functions of R into R such that $g(0) > 0$, $h(0) > 0$ and $W_{(g,h)}$ is an open subset of R^2 containing the origin with g, h are positive on a neighborhood of zero. Let $g_+(h_+)$ be the smallest positive root of $g(x) = 0\,(h(x) = 0$ resp.) if any, or otherwise $g_+ = +\infty(h_+ = +\infty$resp.), and $g_-(h_-)$ be the greatest negative root of $g(x) = 0\,(h(x) = 0$ resp.) if any, or otherwise $g_- = -\infty(h_- = -\infty$resp.). It is easily verified that the level curves of $V_{(g,h)}$ are closed curves surrounding the origin $(0, 0)$ and contained in $W_{(g,h)}$. Moreover, $V_{(g,h)}(x, 0)$, $V_{(g,h)}(0, y)$ are strictly increasing on $(0, g_+)$, $(0, h_+)$, respectively and strictly decreasing on $(g_-, 0)$, $(h_-, 0)$, respectively. If $V_{(g,h)}(x, y)$ is a first integral of the system (1.1) and the origin is a center, then the derivative of $V_{(g,h)}$ relative to system (1.1) is zero and hence every nontrivial solution starting from $(0, g_+)$ in the x-axis is periodic. The largest neighborhood of $(0, 0)$ which is entirely covered by periodic orbits is called the period annulus of $(0, 0)$, and we will denote it by Ω_0. A center is said to be a global center when its period annulus is the whole plane, means when $\Omega_0 = R^2$. A center is said to be nondegenerate when the linearized vector field at the critical point has two nonzero eigenvalues. It is well known that only nondegenerate centers can be isochronous – for more details see [3,5,8]. When $P(x, y)$, $Q(x, y)$ in system (1.1) are analytic it implies that the period annulus of an isochronous center is unbounded. It is clear that $V_{(g,h)}(x, y) > 0$ for every $(x, y) \in W_{(g,h)}\backslash\{(0, 0)\}$, moreover, $W_{(g,h)}(\Omega_0) = [0, E_*)$, where $E_* \in R^+ \cup \{+\infty\}$. The set of all periodic orbits in the period annulus can be parametrized by the energy E. Thus for each $E \in (0, E_*)$ we will denote the periodic orbit in Ω_0 of energy level E by γ_E. This allows us to consider the period function over $(0, E_*)$ instead of the original period function which is defined over the set of periodic orbits contained in the period annulus. Therefore, later in this article we will deal with the period function $T(E)$ which assigns to each periodic orbit γ_E of energy $E \in (0, E_*)$ its period.

In Section 2, we first consider the system (1.1) and give a sufficient condition of functional type in order for the origin to be center and this is given in Proposition 1. Then we deduce a sufficient condition for the center of systems of ODE (2.3) reducible to the equation $\ddot{x} = f(x, \dot{x})$ given in Corollary 1. We also give a sufficient condition for the monotonicity of the associated period function and the isochronicity of center of system (2.3) given by Theorem 1. In Section 3, we introduce three examples exhibiting applications of the results. In Example 1, we apply Proposition 1 in order to construct a rational differential system with bounded period annulus whose boundary has no singular points. In Example 2, we apply Corollary 1 and Theorem 1 to the equation of Lienard type (3.3) with the condition $f(x) = \alpha g(x)$, for some $\alpha \in R$. In Example 3, we consider a particular system that interests specially the physicists.

2. The main results

Proposition 1. *If there are two continuous functions h and g with $h(0)g(0) \neq 0$ such that*

$$xP(x,y)h(y) + \delta yQ(x,y)g(x) = 0 \tag{2.1}$$

in a neighborhood of $(0, 0)$, *where*

$$\delta = \begin{cases} +1 & \text{if } h(0)g(0) > 0 \\ -1 & \text{if } h(0)g(0) < 0. \end{cases}$$

Then the origin $(0, 0)$ *is a center for the system* (1.1).

Proof. We assume, without loss of generality, that $g(0) > 0$, and $h(0) > 0$.

Consider the positive function

$$V_{(g,h)} : W_{(g,h)} \to R$$

as defined in (1.2).

In case of $g(0) < 0$ (or $h(0) < 0$) we replace $g(s)$ with $-g(s)$ (or $h(t)$ with $-h(t)$), respectively in the definition of $V_{(g,h)}(x, y)$. Let

$$G(x) = \int_0^x \frac{s}{g(s)}\mathrm{d}s, \qquad H(y) = \int_0^y \frac{t}{h(t)}\mathrm{d}t \tag{2.2}$$

then $V_{(g,h)}(x, y) = G(x)+H(y)$. The functions G and $H$are of class C^1, strictly increasing on $[0, g_+)$, $[0, h_+)$, respectively, and strictly decreasing on $(g_-, 0]$, $(h_-, 0]$, respectively, $G(x)$ has a quadratic minimum at the origin, where $g_\pm$, $h_\pm$ are as defined in Section 1. Therefore the level curves of $V_{(g,h)}(x, y)$ are all closed curves surrounding the origin. Moreover, the derivative of $V_{(g,h)}$ relative to system (1.1) is

$$\dot{V}_{(g,h)}(x, y) = \frac{xP(x, y)h(y) + yQ(x, y)g(x)}{g(x)h(y)}.$$

From the hypothesis, we get $\dot{V}_{(g,h)}(x, y) = 0$ in a neighborhood of $(0, 0)$. Therefore, every nontrivial solution starting from $(0, g_+)$ in the x-axis is periodic orbit surrounding the origin. Hence the origin is a center. □

We will study the monotonicity of the period function of system (1.1) assuming that the system possesses a center at the origin. In order to find the period function, recall from the hypothesis of the Proposition 1, that

$$xh(y)\frac{\mathrm{d}x}{\mathrm{d}t} + yg(x)Q(x, y) = 0$$

in a neighborhood of the origin. Since each trajectory of the vector field $P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$ is a level curve of $V_{(g,h)}(x, y)$, then the period of the orbit γ_E with energy $E \in (0, E_*)$, is given by the integral of $\mathrm{d}t$ along this closed orbit in clockwise orientation or counterclockwise orientation according to the direction of the flow of the trajectories of the vector field of the system. Then

$$\begin{aligned} T &= \oint_{\gamma_E} \mathrm{d}t \\ &= \int_{\gamma_E^+} \mathrm{d}t + \int_{\gamma_E^-} \mathrm{d}t \end{aligned}$$

where γ_E^+ is the part of the closed orbit γ_E which is above the x-axis, γ_E^- is the part of the closed orbit γ_E which is below the x-axis. Therefore

$$T = \int_{\gamma_E^+} \frac{-xh(y)}{yg(x)Q(x, y)}\mathrm{d}x + \int_{\gamma_E^-} \frac{-xh(y)}{yg(x)Q(x, y)}\mathrm{d}x$$

where the equation of the curve of this orbit γ_E, is given by $V_{(g,h)}(x, y) = E \in (0, E_*)$. This integral form of Tmust be expressed through a suitable change of variables so that we can compute $\frac{\mathrm{d}T}{\mathrm{d}E}$, from which we can decide if $T$is monotone increasing or monotone decreasing on some interval $(0, E_*)$.

Now we apply the above method to introduce some center conditions then we study the monotonicity of the period function of the center of systems of differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x, y) \end{aligned} \tag{2.3}$$

which is reducible to the equation

$$\ddot{x} = f(x, \dot{x}) \tag{2.4}$$

whereas $f(0, 0) = 0$.

Corollary 1. *Suppose that $f \in C^1$, $xf(x, 0) < 0$ in a punctured neighborhood of zero I_0 and the term $\frac{f(x,y)}{f(x,0)}$ is independent of x. Then the origin $(0, 0)$ is a center for system (2.3).*

Proof. Since $\frac{f(x,y)}{f(x,0)}$ is independent of x, then $\frac{f(1,y)}{f(1,0)} = \frac{f(x,y)}{f(x,0)}$ for all $(x, y) \in I_0 \times R$. Choose

$$h(y) = \frac{f(1, y)}{f(1, 0)}. \tag{2.5}$$

It is clear that h is continuous and $h(y) > 0$, in a neighborhood of zero with $h(0) = 1$.

Choose

$$g(x) = \frac{-x}{f(x, 0)}.$$

Since $xf(x, 0) < 0$ for $x \neq 0$ and $f \in C^1$ then $g(x) > 0$ in a neighborhood of zero.

Hence $h(0)g(0) > 0$, then $\delta = 1$ and

$$xyh(y) - yf(x, y)\frac{x}{f(x, 0)} = xy\frac{f(1, y)}{f(1, 0)} - xy\frac{f(x, y)}{f(x, 0)}$$

and this is equal to zero, since $\frac{f(1,y)}{f(1,0)} = \frac{f(x,y)}{f(x,0)}$.

Therefore, the hypothesis of Proposition 1 is satisfied, and hence the origin $(0, 0)$ is a center for system (2.3). □

Notice that, if we put

$$f_2(y) = \frac{f(x,y)}{f(x,0)}, \quad \text{and} \quad f_1(x) = f(x,0), \quad \text{then}\, f(x,y) = f_1(x)f_2(y),$$

then from the Corollary we can say that, if $f(x,y) = f_1(x)f_2(y)$ is separable function and $xf_1(x)f_2(0) < 0$ for $x \neq 0$, then $(0,0)$ is a center.

Sufficient conditions for the monotonicity of the associated period function and the isochronicity of center of system (2.3) are given by the following Theorem 1. We consider a center of system (2.3) at the origin surrounded by periodic orbits each of which is a level energy curve of $V(x,y)$ which is defined with energy $E \in (0, E_*)$, as follows

$$V(x,y) = -\int_0^x f(s,0)\mathrm{d}s + \int_0^y \frac{f(1,0)t}{f(1,t)}\mathrm{d}t \tag{2.6}$$

for $x \in I_0 \cup \{0\}$ and $y \in I_f = (f_-, f_+)$ the largest open interval containing zero on which $f(0,y) > 0$, where $f_- \in R^+ \cup \{-\infty\}, f_+ \in R^+ \cup \{+\infty\}$.

Let us set

$$L(x) = (u'^2(x) - u(x)u''(x))\left(\frac{1}{P(x)} - \frac{1}{N(x)}\right) - 2u'^2(x)(E - G(x))\left(\frac{Z_+(x)}{P^2(x)} - \frac{Z_-(x)}{N^2(x)}\right) \tag{2.7}$$

where

$$G(x) = -\int_0^x f(s,0)\mathrm{d}s, \qquad H(y) = \int_0^y \frac{t}{F(t)}\mathrm{d}t, \qquad F(t) = \frac{f(1,t)}{f(1,0)} \tag{2.8}$$

$$\text{and } u(x) = \begin{cases} \operatorname{sign}(x)\sqrt{G(x)} & x \in I_0 \\ 0 & x = 0 \end{cases},$$

where I_0 is a deleted neighborhood of *zero*,

$$P(x) = H_+^{-1}(E - G(x)),$$
$$N(x) = H_-^{-1}(E - G(x)),$$
$$Z_\pm(x) = \left.\frac{\mathrm{d}H_\pm^{-1}(z)}{\mathrm{d}z}\right|_{z=E-G(x)},$$

where

H_-, H_+ are the restrictions of H on, $[y_0, 0], [0, y_1]$, respectively

and y_0, y_1 are the lower and upper intersections of the level curve $V(x,y) = E \in (0, E_*)$ with the y-axis.

Remark 1.

1. $\frac{\mathrm{d}P(x)}{\mathrm{d}E} = Z_+(x)\cos^2\theta.$
2. $\frac{\mathrm{d}N(x)}{\mathrm{d}E} = Z_-(x)\cos^2\theta.$
3. $\frac{\mathrm{d}u'(x)}{\mathrm{d}E} = \frac{u''(x)\sin\theta}{2\sqrt{E}u'(x)}.$

Proof. (1) Set $z = E - G(x)$, then from the definition of $P(x)$ we get

$$\begin{aligned}\frac{\mathrm{d}P(x)}{\mathrm{d}E} &= \left.\frac{\mathrm{d}H_+^{-1}(z)}{\mathrm{d}E}\right|_{z=E-G(x)} \\ &= \left.\frac{\mathrm{d}H_+^{-1}(z)}{\mathrm{d}z}\right|_{z=E-G(x)} \frac{\mathrm{d}z}{\mathrm{d}E} \\ &= Z_+(x)\left(1 - \frac{\mathrm{d}G(x)}{\mathrm{d}E}\right).\end{aligned}$$

But $r = u(x) = \operatorname{sign}(x)\sqrt{G(x)}$ and $r = \sqrt{E}\sin\theta$, then $\operatorname{sign}(x)\sqrt{G(x)} = \sqrt{E}\sin\theta$, from which we obtain $\frac{\mathrm{d}G(x)}{\mathrm{d}E} = \sin^2\theta$. Therefore $\frac{\mathrm{d}P(x)}{\mathrm{d}E} = Z_+(x)\cos^2\theta$.

(2) It is similarly proved.
(3)

$$\begin{aligned}\frac{\mathrm{d}u'(x)}{\mathrm{d}E} &= \frac{\mathrm{d}u'(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}E}\\ &= u''(x)\frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}E}\\ &= \frac{u''(x)}{u'(x)}\frac{\sin\theta}{2\sqrt{E}}. \quad \square\end{aligned}$$

In order to state the following Theorem we let

$$I(E_*) = \{x \in R\setminus \{0\} : V(x) \le E_*\}.$$

Theorem 1. *Suppose that $f \in C^1$, $xf(x,0) < 0$ in a punctured neighborhood of zero I_0, and the origin of system (2.3) is center. Then we have*
If $L(x) \ge 0$ on I_0, then the period function is increasing on $(0, E_*)$.
If $L(x) \le 0$ on I_0, then the period function is decreasing on $(0, E_*)$.
If $L(x) \equiv 0$ in a neighborhood of the origin, then the origin is an isochronous center.

Proof. Consider the level curves $(V(x,y) =)\ G(x) + H(y) = E$, for $E \in (0, E_*)$, and the period function

$$T : (0, E_*) \to R$$

where $G(x), H(y)$ are as defined in (2.8). Since $\dot{x} = y$, then $\mathrm{d}t = \frac{\mathrm{d}x}{y}$, and hence the period of the closed orbit γ_E will be given by the integral of $\mathrm{d}t$ along the closed orbit in clockwise orientation, so

$$\begin{aligned}T &= \oint_{\gamma_E} \mathrm{d}t\\ &= \int_{\gamma_E^+} \mathrm{d}t + \int_{\gamma_E^-} \mathrm{d}t\end{aligned}$$

where γ_E^+ is the part of the closed orbit γ_E which is above the x-axis that is given by the equation $y = P(x)$, and γ_E^- is the part of the closed orbit γ_E which is below the x-axis and given by the equation $y = N(x)$.

Let x_0, x_1 be the left and right intersections of the orbit γ_E with the x-axis. Hence

$$\begin{aligned}T &= \int_{x_0}^{x_1} \frac{\mathrm{d}x}{P(x)} + \int_{x_1}^{x_0} \frac{\mathrm{d}x}{N(x)}\\ &= \int_{x_0}^{x_1} \frac{N(x) - P(x)}{P(x)N(x)}\mathrm{d}x.\end{aligned}$$

Since $G(x)$ has a quadratic minimum at the origin, then $u(x)$ is smooth in the interval (x_0, x_1). Moreover, it is easy to see that $u'(x) > 0$, so u is monotone increasing on (x_0, x_1). In order to express the integral so that we can compute $\frac{\mathrm{d}T}{\mathrm{d}E}$ we consider the change of variables $r = u(x)$ in the integrand to obtain

$$T = \int_{-\sqrt{E}}^{+\sqrt{E}} \frac{N(x) - P(x)}{P(x)N(x)} \frac{\mathrm{d}r}{u'(x)}$$

where $x = u^{-1}(r)$. Use the change of variables, $r = \sqrt{E}\sin\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we get,

$$T = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{N(x) - P(x)}{P(x)N(x)} \frac{\sqrt{E}\cos\theta}{u'(x)}\mathrm{d}\theta$$

where $x = u^{-1}(\sqrt{E}\sin\theta)$.

It is clear from the definitions of N, P and u' to see that the integrand of the above integral is differentiable with respect to $E \in (0, E_*)$, so we can differentiate with the integral in the above form to obtain,

$$\frac{\mathrm{d}T}{\mathrm{d}E} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}}{\mathrm{d}E}\left[\frac{N(x) - P(x)}{P(x)N(x)} \frac{\sqrt{E}\cos\theta}{u'(x)}\right]\mathrm{d}\theta.$$

Simplify the term

$$\frac{\mathrm{d}}{\mathrm{d}E}\left[\frac{N(x)-P(x)}{P(x)N(x)}\frac{\sqrt{E}\cos\theta}{u'(x)}\right]$$

with direct computation recalling that

$$\begin{aligned} u(x) &= \operatorname{sign}(x)\sqrt{G(x)}, \quad \text{for } x\in I_0 \text{ and } \textit{zero} \text{ for } x=0,\\ u'(x) &= \operatorname{sign}(x)\frac{G'(x)}{2\sqrt{G(x)}}, \quad \text{for } x\in I_0\\ u''(x) &= \operatorname{sign}(x)\frac{2GG''-G'^2}{4G^{\frac{3}{2}}}, \quad \text{for } x\in I_0 \end{aligned} \tag{2.9}$$

and the terms that are given in the Remark 1, we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}E}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\theta}{2\sqrt{E}u'^3}L(x)\mathrm{d}\theta$$

where $L(x)$ is the expression given in (2.7). It is clear that $\frac{\cos\theta}{2\sqrt{E}u'^3(x)}\geq 0$ on the interval of integration, so we conclude that if $L(x)\geq 0$ then $\frac{\mathrm{d}T}{\mathrm{d}E}\geq 0$ and hence T is increasing, if $L(x)\leq 0$ then $\frac{\mathrm{d}T}{\mathrm{d}E}\leq 0$ and hence T is decreasing, and if $L(x)\equiv 0$ then T is constant and hence the origin is an isochronous center. □

It is perhaps remarkable that in the proof of the theorem we obtained

$$\frac{\mathrm{d}T}{\mathrm{d}E}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\theta}{2\sqrt{E}u'^3}L(x)\mathrm{d}\theta$$

and this gives both sufficient and necessary conditions for monotonicity of period function and isochronicity of the center. But from computation point of view it may be difficult to compute the integration of the term $\frac{\cos\theta}{2\sqrt{E}u'^3}L(x)$. For that we prefer to introduce a sufficient condition only.

In order to state the following Corollary we set,

$$M(x)=\left(1-\frac{(E-G(x))Z_+(x)}{P(x)}\right)f^2(x,0)+G(x)\frac{\mathrm{d}f(x,0)}{\mathrm{d}x}. \tag{2.10}$$

Corollary 2. *Suppose that $f(x,-y)=f(x,y)$, we have*
If $M(x)\geq 0$ on I_0, then the period function is increasing on $(0,E_*)$.
If $M(x)\leq 0$ on I_0, then the period function is decreasing on $(0,E_*)$.
If $M(x)\equiv 0$ in a neighborhood of the origin, then the origin is an isochronous center.

Proof. If $f(x,-y)=f(x,y)$ then the expression

$$\frac{yf(x,0)}{f(x,y)}\left(=\frac{y}{F(y)}\right)$$

is an odd function in y, and then

$$H(y)=\int_0^y\frac{t}{F(t)}\mathrm{d}t$$

is even function. Therefore $-H_-^{-1}=H_+^{-1}=P=-N$ and consequently $\frac{Z_+(x)}{P^2(x)}=-\frac{Z_-(x)}{N^2(x)}$. Substituting these terms in (2.7) one obtains

$$L(x)=\frac{2(u'^2-u''u)}{P(x)}-\frac{4u'^2(E-G(x))Z_+(x)}{P^2(x)}. \tag{2.11}$$

From definitions of $u(x)$ and $G(x)$ in (2.8) we get

$$\begin{aligned} u(x) &= \operatorname{sign}(x)\sqrt{G(x)},\ u'(x)=\operatorname{sign}(x)\frac{-f(x,0)}{2\sqrt{G(x)}},\\ u''(x) &= \operatorname{sign}(x)\frac{-2G(x)\frac{\mathrm{d}f(x,0)}{\mathrm{d}x}-f^2(x,0)}{4G^{\frac{3}{2}}(x)} \end{aligned}$$

and substitute in (2.11) we get

$$L(x) = \frac{1}{P(x)G(x)}\left[\left(1 - \frac{(E-G(x))Z_+(x)}{P(x)}\right)f^2(x,0) + G(x)\frac{\mathrm{d}f(x,0)}{\mathrm{d}x}\right]. \tag{2.12}$$

Since the term $P(x)G(x)$ is positive, then $L(x)$ and $M(x)$ have the same sign. Therefore with applying Theorem 1 we get the conclusion of the corollary. □

3. Applications

3.1. Example 1

In this example we apply Proposition 1 in order to construct a differential system with period annulus whose boundary has no singular point.

Consider the differential system

$$\begin{aligned} \dot{x} &= \frac{yH(x,y)}{h(y)} \\ \dot{y} &= \frac{-xH(x,y)}{g(x)} \end{aligned} \tag{3.1}$$

where H is analytic function on an open set containing $(0,0)$, h, g are analytic functions on an open interval containing zero with $g(0)h(0) \neq 0$, and the system has no singular point in a neighborhood of the origin except the origin $(0,0)$. It is clear that the condition (2.1) is satisfied then from Proposition 1 one concludes that system (3.1) has center at the origin and the period annulus of $(0,0)$ is an open set U_0. It is clear that the boundary of the period annulus has no singular point. The intersection of U_0 with the x-axis is the interval $I_g = (g_-, g_+)$, and the intersection of U_0 with the y-axis is the interval $I_h = (h_-, h_+)$, where $g_\pm$ and $h_\pm$ are as defined in the section 1. If $I_g = R$, and $I_h = R$, and H is analytic on the entire space R^2 and the system has no singular point except at the origin then the origin is global center. Consider as a concrete example the rational differential system

$$\begin{aligned} \dot{x} &= \frac{x^2y + y^3 + y}{1-y^2} \\ \dot{y} &= \frac{-x^3 - xy^2 - x}{1-x^2}. \end{aligned} \tag{3.2}$$

Here $H(x,y) = x^2+y^2+1,\ g(x) = 1-x^2, h(y) = 1-y^2, g_+ = 1, g_- = -1, h_+ = 1,\ h_- = -1$. Following the formula (1.2) we obtain that the Hamiltonian of the system (3.2) will be of the form $\ln \frac{1}{\sqrt{(1-x^2)(1-y^2)}}$, but for simplicity we may assume without loss of generality that

$$V(x,y) = \ln \frac{1}{(1-x^2)(1-y^2)}$$

for $|x| < 1$, $|y| < 1$. The origin is a center and the period annulus is bounded contained in the square $|x| < 1$, $|y| < 1$, and the system has only one singular point which is the origin. This means the boundary of the period annulus has no singular point.

3.2. Example 2

Consider the equations of the type

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0 \tag{3.3}$$

with $f(x), g(x) \in C^1$, $xg(x) > 0$, for $x \neq 0$. These equations have many physical applications, see [10]. Indeed, that is a model of one dimensional oscillator studied at the classical and also at the quantum level. We will study, under certain hypothesis, the monotonicity character of the period function by means of an appropriate equivalent differential system. These Eq. (3.3) are considered in [4,16] as well to study their qualitative behavior from the point of view of solution and periodicity of the solutions. Here we apply our Theorems to establish center character and the periodicity of the solutions of the system.

We discuss the case that if there is a real number $\alpha \neq 0$ such that $f(x) = \alpha g(x)$ on some interval I containing the origin, then we apply Corollary 1 and Theorem 2. An important particular case of this system is discussed in Example 3. If $\alpha = 0$, the equation will be $\ddot{x} + g(x) = 0$ which is extensively studied by many authors, so we exclude $\alpha = 0$.

Consider the equivalent system of the form,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\alpha g(x) y^2 - g(x). \end{aligned} \tag{3.4}$$

Proposition 2. *If $g \in C^1$, $xg(x) > 0$, for $x \neq 0$, $\alpha \neq 0$. Then the origin of the system (3.4) is a center.*

Proof. We have

$$f(x,y) = -(\alpha y^2 + 1) g(x).$$

Therefore $f \in C^1$, $xf(x,0) < 0$ for $x \neq 0$ and $\frac{f(x,y)}{f(x,0)} = \alpha y^2 + 1$ is independent of x. Therefore, by Corollary 1, the origin $(0,0)$ is a center. □

Since $F(t) = \alpha t^2 + 1$, we notice that in the above Proposition the region of the periodic orbits of energy E is contained in the strip $S = \left\{(x,y) \in R^2 : y_0 < y < y_1\right\}$ where y_0, y_1 are the roots of $\alpha y^2 + 1 = 0$, hence for $\alpha > 0$ we have $y_0 = -\infty$, $y_1 = +\infty$, and for $\alpha < 0$, we have $y_0 = -\sqrt{\frac{-1}{\alpha}}$, $y_1 = \sqrt{\frac{-1}{\alpha}}$.

In order to state a sufficient condition for monotonicity of the period function and isochronicity of the center of system (3.4) we consider a level energy curve of $V(x,y)$ which is defined in (2.6) with energy $E \in (0, E_*)$. Let us set,

$$Q(x) = \left(1 - \frac{\alpha^2 (E-G)}{1 - \mathrm{e}^{2\alpha(G-E)}}\right) g^2(x) - g'(x) G(x) \tag{3.5}$$

where $G(x) = \int_0^x g(s)\mathrm{d}s$.

Proposition 3. *Let $g(x) \in C^1(a,b)$, for some $a < 0 < b$, with $xg(x) > 0$ for $x \in J_0 = (a,b)\backslash\{0\}$ in system (3.4). Then we have,*
If $Q(x) \geq 0$ on J_0, then the period function is increasing on $(0, E_*)$.
If $Q(x) \leq 0$ on J_0, then the period function is decreasing on $(0, E_*)$.
If $Q(x) \equiv 0$ in a neighborhood of the origin, then the origin is an isochronous center.

Proof. Here we have,

$$\begin{aligned} f(x,y) &= -(\alpha y^2 + 1) g(x), \\ f(x,0) &= -g(x), \\ G(x) &= \int_0^x g(s)\mathrm{d}s, \\ F(t) &= \alpha t^2 + 1, \\ H(y) &= \frac{1}{2\alpha} \ln(1 + \alpha y^2), \\ Z_+(x) &= \frac{\alpha^{\frac{3}{2}} \mathrm{e}^{2\alpha(E-G(x))}}{\sqrt{\mathrm{e}^{2\alpha(E-G(x))} - 1}}, \\ u(x) &= \mathrm{sign}(x)\sqrt{G(x)} \quad \text{for } x \in I_0 \text{ and } \textit{zero} \text{ for } x = 0. \end{aligned}$$

It is clear that $f(x,-y) = f(x,y)$, so we can apply Corollary 2. Substitute these terms in the formula (2.10), simplify with direct computations, we obtain that the expression of $M(x)$ given in (2.10) will be in the form (3.5). Therefore with applying the Corollary 2 we get the conclusion of the Proposition. □

3.3. Example 3

We consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{-\alpha^2 x}{1 + \lambda x^2} + \frac{\lambda x y^2}{1 + \lambda x^2} \end{aligned} \tag{3.6}$$

with $\lambda > 0$, $\alpha \neq 0$. This system is a particular case of Eq. (3.3) and it interests specially the physicists. The general solution of (3.6) takes the form $x(t) = E\sin(\frac{2\pi}{T(E)} t + \phi)$. Curiously, that is the only case which permits explicitly determination of the amplitude dependence of the period function

$$T(E) = \frac{2\pi}{\alpha}\sqrt{1 + \lambda E^2}.$$

$T(0) = \frac{2\pi}{\alpha}$ corresponds to the center 0. It is clear that $T(E)$ is an increasing function. We take this system in order to show the applicability of our Theorem 1.

Notice $f(x, y) = \frac{-\alpha^2 x}{1+\lambda x^2} + \frac{\lambda x y^2}{1+\lambda x^2}$, so $f \in C^1$, $xf(x, 0) < 0$ for $x \neq 0$, f is separable and $f(x, -y) = f(x, y)$, therefore we can apply Corollary 2.

We have

$$f(x, 0) = \frac{-\alpha^2 x}{1 + \lambda x^2}$$

Here $F(t) = 1 - \frac{\lambda}{\alpha^2} t^2$, with $I_F = (\frac{-\alpha}{\sqrt{\lambda}}, \frac{\alpha}{\sqrt{\lambda}})$. Following the formula (1.2) we obtain that the corresponding Hamiltonian is $\frac{\alpha^2}{2\lambda}\left[\ln(1 + \lambda x^2) - \ln(1 - \frac{\lambda}{\alpha^2} y^2)\right]$. We can assume, without loss of generality, that

$$V(x, y) = \ln(1 + \lambda x^2) - \ln\left(1 - \frac{\lambda}{\alpha^2} y^2\right)$$

with

$$\begin{aligned} G(x) &= \ln(1 + \lambda x^2) \\ &= \frac{-2\lambda}{\alpha^2}\int_0^x f(s, 0)\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} H(y) &= -\ln\left(1 - \frac{\lambda}{\alpha^2} y^2\right) \quad \text{for } y \in I_F \\ &= \frac{2\lambda}{\alpha^2}\int_0^y \frac{tf(1, 0)}{f(1, t)}\mathrm{d}t \\ &= \frac{2\lambda}{\alpha^2}\int_0^y \frac{t}{F(t)}\mathrm{d}t, \\ H_+^{-1}(z) &= \frac{\alpha}{\sqrt{\lambda}}\sqrt{1 - \mathrm{e}^{-z}} \quad \text{for } 0 \le z \le \frac{\alpha}{\sqrt{\lambda}}, \\ P(x) &= \frac{\alpha}{\sqrt{\lambda}}\sqrt{1 - \mathrm{e}^{G(x) - E}} \\ &= \frac{\alpha}{\sqrt{\lambda}}\sqrt{1 - \frac{1 + \lambda x^2}{\mathrm{e}^E}} \quad \text{for } -\sqrt{\frac{\mathrm{e}^E - 1}{\lambda}} \le x \le \sqrt{\frac{\mathrm{e}^E - 1}{\lambda}} \\ Z_+(x) &= \frac{\alpha^2 \mathrm{e}^{G(x) - E}}{2\lambda P(x)} \\ &= \frac{\alpha(1 + \lambda x^2)}{2\sqrt{\lambda}\sqrt{\mathrm{e}^{2E} - 1 - \lambda x^2}}, \quad \text{for } -\sqrt{\frac{\mathrm{e}^E - 1}{\lambda}} \le x \le \sqrt{\frac{\mathrm{e}^E - 1}{\lambda}} \\ u(x) &= \operatorname{sign}(x)\sqrt{\ln(1 + \lambda x^2)} \quad \text{for } x \in I_0 \text{ and } \textit{zero} \text{ for } x = 0, \\ u'(x) &= \frac{\lambda x}{(1 + \lambda x^2) u(x)} \quad \text{for } x \in I_0, \\ u''(x) &= \frac{\lambda(1 - \lambda x^2)\ln(1 + \lambda x^2) - \lambda x^2}{(1 + \lambda x^2)^2\left[\ln(1 + \lambda x^2)\right]^{\frac{3}{2}} u(x)} \quad \text{for } x \in I_0. \end{aligned}$$

Since both $G(x)$ and $H(y)$ are even functions then the closed curve $G(x) + H(y) = E$ is an oscillating solution surrounding the origin and this closed orbit is symmetric with both *axes*. Substitute these terms in the formula (2.10), simplify with direct computation we obtain the form

$$\frac{1}{2}\alpha^2 x^2\left(2 + \frac{E - G(x)}{\mathrm{e}^{E - G(x)} - 1}\right) + (1 - \lambda x^2)\ln(1 + \lambda x^2). \tag{3.7}$$

Since $E \ge G(x)$, it is clear that $\frac{E - G(x)}{\mathrm{e}^{E - G(x)} - 1} \ge 0$, therefore the term (3.7) is positive for $|x| \le \frac{1}{\sqrt{\lambda}}$. From Corollary 2 we conclude that the period function is increasing.

Acknowledgement

The author thanks the University of Sharjah for its support.

References

[1] K.I. Al-Dosary, The monotonicity of the period function for $\ddot{x} + f_2(x)\dot{x}^2 + f_1(x)\dot{x} + g(x) = 0$, Int. J. Pure Appl. Math. 32 (2) (2006) 167–189.
[2] K.I. Al-Dosary, Sufficient conditions for oscillation of second order differential equations (in press).
[3] C. Chicone, The monotonicity of the period function for planar Hamiltonian vector fields, J. Differential Equations 69 (1987) 310–321.
[4] A.R. Chouikha, Isochronicity centers of Lienard type equations and applications, J. Math. Anal. Appl. (in press).
[5] A.R. Chouikha, V. Romanovski, X. Chen, Isochronicity of analytic systems via Urabek's criterion, J. Phys. A 40 (2007) 2313–2327.
[6] C.J. Christopher, J. Devlin, Isochronous Centers in Planar Polynomial Systems, vol. 28 no. 1 1997 pp. 162–177.
[7] A. Cima, A. Gasull, F. Manosas, Period function for Hamiltonian systems, J. Differential Equations 168 (2000) 180–199.
[8] A. Cima, F. Manosas, J. Villadelprat, Isochronicity for several classes of Hamiltonian systems, J. Differential Equations 157 (1999) 373–413.
[9] A. Gasull, V. Guillamon and Manosa, An explicit expression of the first Lyapunov and period constants with applications, J. Math. Anal. Appl. 211 (1997) 190–212.
[10] M. Lakshmanan, S. Raghasekar, Nonlinear dynamics, in: Integrability, Chaos and Patterns, Advanced Texts in Physics, Springer-Verlag, Berlin, 2003.
[11] W.S. Loud, Behavior of the period of solutions of certain planar autonomous systems near centers, Contrb. Differential Equations 3 (1964) 21–36.
[12] P. Mardesic, C. Rousseau, B. Toni, Linearization of isochronous centers, J. Differential Equations 121 (1995) 67–108.
[13] V.V. Nemytskii, V.V. Stepanov, Qualitative Theory of Differential Equations, Priceton Univ. Press, Princeton, NJ, 1960.
[14] M. Sabatini, On the period function of Lienard system, J. Differential Equations 152 (1999) 467–487.
[15] M. Sabatini, Characterizing isochronous centers by Lie brackets, Differential Equations Dynam. Systems 5 (1997) 91–99.
[16] M. Sabatini, On the period function of $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$, J. Differential Equations 196 (2004) 151–168.
[17] M. Villarini, Regularity properties of the period function near a center of a planar vector field, Nonlinear Anal. 19 (1992) 787–803.